A Complex-Analytic Representation of Rounding Functions

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July 18, 2025

Abstract

We present a single closed-form expression that reproduces the standard round-to-nearest integer function on the real line and extends naturally to the complex plane. The formula

$$\text{Round}(z) = z + \frac{\ln(e^{2\pi i z})}{2\pi} \cdot i$$

where ln denotes the principal branch logarithm, maps each vertical strip $k - \frac{1}{2} < \text{Re}(z) \leq k + \frac{1}{2}$ holomorphically to the integer k. The branch cut of the logarithm coincides precisely with the classical half-integer tie points where the round-to-nearest function is discontinuous. We prove the equivalence on the real axis, analyze the complex extension, and demonstrate how repositioning the branch cut generates other rounding variants through the same analytic framework.

1 Introduction

The round-to-nearest integer function is traditionally defined as the piecewise expression

$$x \mapsto \lfloor x + \frac{1}{2} \rfloor \tag{1}$$

which exhibits discontinuities at half-integer values and resists smooth analytic treatment.

In this paper, we show that the same rounding behavior can be captured by a closed-form, complex-analytic formula

$$\text{Round}(z) = z + \frac{\ln(e^{2\pi i z})}{2\pi} \cdot i \tag{2}$$

where ln denotes the logarithm on the principal branch. This representation has several interesting properties: it reproduces the classical round-to-nearest function on real inputs, extends naturally to the complex plane, and unifies multiple rounding modes through branch cut manipulation.

The key insight is that the discontinuities of the rounding function correspond exactly to the branch cut of the complex logarithm. By shifting this branch cut, we can transform between different rounding behaviors—including floor and ceiling rounding—without altering the underlying analytic structure.

Our main contributions are: (1) a proof that formula (2) reproduces standard rounding on $\mathbb{R}$, (2) an analysis of its complex extension showing that vertical strips collapse to integers, and (3) a complete characterization of how branch cut variations generate different rounding modes.

2 Real Axis Analysis

We begin by establishing that our complex-analytic formula reproduces the standard rounding function on real inputs.

Theorem 2.1. *For every real number $x \notin \mathbb{Z} + \frac{1}{2}$,*

$$\text{Round}(x) = \left\lfloor x + \frac{1}{2} \right\rfloor. \tag{3}$$

Proof. Let $x \in \mathbb{R} \setminus (\mathbb{Z} + \frac{1}{2})$ and write $e^{2\pi ix} = e^{i\theta}$ where θ is the argument of the complex exponential. Since x is not a half-integer, the point $e^{2\pi ix}$ does not lie on the negative real axis, so the principal branch logarithm is well-defined.

Define $k = \lfloor x + \frac{1}{2} \rfloor$. Then $x \in (k - \frac{1}{2}, k + \frac{1}{2})$, which implies $2\pi x \in (2\pi k - \pi, 2\pi k + \pi)$. We can therefore write

$$\theta = 2\pi x - 2\pi k \in (-\pi, \pi), \tag{4}$$

placing θ in the principal argument range.

Since $\ln(e^{i\theta}) = i\theta$ on the principal branch, we have

$$\ln(e^{2\pi ix}) = i(2\pi x - 2\pi k) = 2\pi i(x - k). \tag{5}$$

Substituting into equation (2) yields

$$\text{Round}(x) = x + \frac{2\pi i(x-k)}{2\pi} \cdot i = x + i^2(x-k) = x - (x-k) = k = \left\lfloor x + \frac{1}{2} \right\rfloor, \tag{6}$$

completing the proof. ∎

Remark 2.2. The function $\text{Round}(x)$ is undefined at half-integers $x \in \mathbb{Z} + \frac{1}{2}$, where $e^{2\pi i x} = -1$ lies on the branch cut of the principal logarithm. This corresponds exactly to the tie points where the classical round-to-nearest function exhibits its discontinuous jumps.

3 Complex Extension

We now analyze the behavior of $\text{Round}(z)$ for complex arguments, showing that it maps vertical strips to integers while maintaining holomorphicity within each strip.

Lemma 3.1. *Let $z = x + iy$ where $x, y \in \mathbb{R}$, and define $k = \lfloor x + \frac{1}{2} \rfloor$. If $\text{Re}(z) \notin \mathbb{Z} + \frac{1}{2}$, then $\text{Round}(z) = k$.*

Proof. We can factor the complex exponential as

$$e^{2\pi i z} = e^{2\pi i (x+iy)} = e^{-2\pi y} \cdot e^{2\pi i x}. \tag{7}$$

Since $x \notin \mathbb{Z} + \frac{1}{2}$, the point $e^{2\pi i x}$ does not lie on the negative real axis. The factor $e^{-2\pi y} > 0$ is real and positive, so $e^{2\pi i z}$ has the same argument as $e^{2\pi i x}$.

Following the argument in Theorem 2.1, we adjust the raw argument $2\pi x$ by subtracting $2\pi k$ to place it in $(-\pi, \pi]$. The principal branch logarithm then gives

$$\ln(e^{2\pi i z}) = \ln(e^{-2\pi y}) + \ln(e^{2\pi i x}) = -2\pi y + 2\pi i (x - k). \tag{8}$$

Substituting into equation (2), we obtain

$$\text{Round}(z) = z + \frac{-2\pi y + 2\pi i (x-k)}{2\pi} \cdot i = x + iy - iy + i^2(x-k) = x - (x-k) = k. \tag{9}$$

∎

Theorem 3.2. *For each integer k, the function $\text{Round}(z)$ is constant and holomorphic on the open vertical strip $\{z \in \mathbb{C} : k - \frac{1}{2} < \text{Re}(z) < k + \frac{1}{2}\}$, where it takes the value k. The function is non-analytic only on the vertical lines $\text{Re}(z) \in \mathbb{Z} + \frac{1}{2}$.*

Proof. The first statement follows immediately from Lemma 3.1 and the fact that constant functions are holomorphic with derivative zero.

For the second statement, note that $\mathrm{Round}(z)$ is undefined when $e^{2\pi iz}$ lies on the negative real axis, which occurs precisely when $\mathrm{Re}(z) \in \mathbb{Z} + \frac{1}{2}$. These are the branch lines of the principal logarithm in our parametrization. □

4 Branch Cut Variants and Rounding Modes

The principal branch logarithm can be replaced with other branch choices to achieve different rounding behaviors. This section provides a complete analysis of how branch cut positioning controls the resulting rounding mode.

Definition 4.1. For any real parameter α, let $\ln_\alpha$ denote the logarithm with argument range $(\alpha - \pi, \alpha + \pi]$. Define the α-parameterized rounding function as

$$\mathrm{Round}_\alpha(z) = z + \frac{\ln_\alpha(e^{2\pi iz})}{2\pi} \cdot i. \tag{10}$$

Theorem 4.2. *The α-parameterized rounding function* Round_α *has the following properties:*

1. ***Discontinuity set**:* Round_α *is discontinuous at z where* $\mathrm{Re}(z) \in \frac{\alpha}{2\pi} - \frac{1}{2} + \mathbb{Z}$.
2. ***Rounding formula**: For real x not in the discontinuity set,*

$$\mathrm{Round}_\alpha(x) = \left\lfloor x - \frac{\alpha}{2\pi} + \frac{1}{2} \right\rfloor. \tag{11}$$

3. ***Special cases**:*

$$\alpha = 0: \quad \mathrm{Round}_0(x) = \left\lfloor x + \frac{1}{2} \right\rfloor \quad \textit{(round-to-nearest)} \tag{12}$$

$$\alpha = \pi: \quad \mathrm{Round}_\pi(x) = \lfloor x \rfloor \quad \textit{(floor function)} \tag{13}$$

$$\alpha = -\pi: \quad \mathrm{Round}_{-\pi}(x) = \lceil x \rceil \quad \textit{(ceiling function)} \tag{14}$$

Proof. **Part 1**: The α-branch logarithm has discontinuities where the argument equals $\alpha - \pi$ or $\alpha + \pi$. For real x, this occurs when $2\pi x \equiv \alpha - \pi \pmod{2\pi}$, giving $x \equiv \frac{\alpha - \pi}{2\pi} \pmod{1}$, or equivalently $x \in \frac{\alpha}{2\pi} - \frac{1}{2} + \mathbb{Z}$.

Part 2: For x not in the discontinuity set, we write $2\pi x = 2\pi k + \theta$ where $k \in \mathbb{Z}$ and $\theta \in (\alpha - \pi, \alpha + \pi]$. This gives $x = k + \frac{\theta}{2\pi}$ with $\frac{\theta}{2\pi} \in \left(\frac{\alpha - \pi}{2\pi}, \frac{\alpha + \pi}{2\pi}\right]$.

Setting $\beta = \frac{\alpha}{2\pi}$, we have $x \in (k + \beta - \frac{1}{2}, k + \beta + \frac{1}{2}]$, so $k = \lfloor x - \beta + \frac{1}{2} \rfloor$.

Since $\ln_\alpha(e^{2\pi ix}) = i\theta = 2\pi i(x-k)$, substitution into equation (10) yields $\text{Round}_\alpha(x) = k = \lfloor x - \frac{\alpha}{2\pi} + \frac{1}{2} \rfloor$.

Part 3: Direct substitution gives the stated special cases. □

Corollary 4.3. *The branch cut parameter α shifts the collapse strips horizontally by $\frac{\alpha}{2\pi}$:*

$$\alpha = 0: \quad \textit{strips} \left(k - \frac{1}{2}, k + \frac{1}{2}\right] \mapsto k \tag{15}$$

$$\alpha = \pi: \quad \textit{strips } (k, k+1] \mapsto k \tag{16}$$

$$\alpha = -\pi: \quad \textit{strips } (k-1, k] \mapsto k \tag{17}$$

Example 4.4. Consider $x = 2.7$:

$$\text{Round}_0(2.7) = \lfloor 2.7 + 0.5 \rfloor = 3 \quad \text{(round to nearest)} \tag{18}$$

$$\text{Round}_\pi(2.7) = \lfloor 2.7 \rfloor = 2 \quad \text{(floor)} \tag{19}$$

$$\text{Round}_{-\pi}(2.7) = \lceil 2.7 \rceil = 3 \quad \text{(ceiling)} \tag{20}$$

The discontinuities occur at different locations: half-integers for $\alpha = 0$, and integers for $\alpha = \pm\pi$.

5 Conclusion

We have shown that the complex-analytic expression (2) provides a unified framework for understanding rounding functions. The key insights are:

1. The discontinuities of rounding functions correspond exactly to branch cuts of the complex logarithm.

2. Different rounding modes (nearest, floor, ceiling) emerge from the same analytic formula through branch cut repositioning.

3. The complex extension reveals the geometric structure: vertical strips in $\mathbb{C}$ collapse holomorphically to integers.

This representation has some practical advantages for certain applications. Unlike piecewise definitions, our formula is trivially differentiable almost everywhere, making it compatible with automatic differentiation systems. The parametric family Round_α provides a systematic way to implement different rounding behaviors within the same analytic framework.

Future work might explore extensions to other special functions, applications to numerical optimization, or higher-dimensional generalizations of the strip-collapse phenomenon.

References

[1] W. Kahan, *Branch cuts for complex elementary functions*, in *The State of the Art in Numerical Analysis*, A. Iserles and M.J.D. Powell, eds., Oxford University Press, 1987, pp. 165–211.